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LOCAL INSTABILITY OF HORIZONTAL TUNNELS OF POLYGONAL SHAPE IN VISCOELASTOPLASTIC MASSES

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The local instability of a horizontal mine tunnel with a regular polygonal cross section in a viscoelastoplastic rock mass is studied within the framework of the exact three-dimensional stability equations. The effect of rock-mass parameters on the critical pressure is estimated.

Key words: *local instability, mine tunnels, viscoelastoplastic medium.*

It is known that the solution of mining engineering problems related to roadway construction and oil- and gas-well drilling reduces to the formulation and solution of problems of the local instability of rock masses in the neighborhood of tunnels under elastoplastic strains [1–4]. This is due to the fact that around tunnels and wells, the stresses at even small depths up to 1 km exceed the rock strength limit, because of which the rock reaches the state of inelastic deformation before the onset of local buckling. The first stage of the solution of this problem consists of finding the stress–strain state of infinite space under dead weight with an infinite cylindrical tunnel having a regular polygonal cross section. The second stage consists of solving the linear problem of stability, i.e., finding the critical pressure distributed uniformly over the tunnel contour. Unlike in [2], in the present paper, we study the local instability of the rock in the near-wellbore zone of a horizontal tunnel with a polygonal cross section using the exact three-dimensional equations of [5]. The properties of the near-wellbore rock are simulated by the relations of a viscoelastoplastic body with translational hardening [6, 7].

In this case, the loading function is given by

$$F = (S_i^j - c(\varepsilon_i^j)^{\rm pl} - \eta(e_i^j)^{\rm pl})(S_j^i - c(\varepsilon_j^i)^{\rm pl} - \eta(e_j^i)^{\rm pl}) - k^2, \tag{1}$$

and the relations of the associate flow law are written as

$$(e_i^j)^{\rm pl} = \lambda(S_i^j - c(\varepsilon_i^j)^{\rm pl} - \eta(e_i^j)^{\rm pl}). \tag{2}$$

Here c is the hardening coefficient, η is the viscosity coefficient, k is the yield strength, $S_i^j = \sigma_i^j - \sigma\delta_i^j$ is the stress tensor deviator, $\sigma = \sigma_k^k/3$, δ_i^j is the Kronecker delta, ε_i^j are the strain tensor components, e_i^j are the strain rate tensor components, and λ is a positive factor.

Investigation of the basic state of a body of volume V characterized by the displacement vector $\mathring{u}_i(x_k, t)$, the stress tensor $\mathring{\sigma}_i^j(x_k, t)$, and the vector of volume $X\mathring{X}_i$ and surface $\mathring{P}_i$ forces reduces to solving a system of variational differential equations subject to appropriate boundary conditions [4].

The equilibrium equations for the regions of plastic ($V^{\rm pl}$) and elastic ($V^{\rm el}$) strains are written as

$$\nabla_i(\sigma_j^i + \mathring{\sigma}_\alpha^i \nabla^\alpha u_j) + X_i - \rho s^2 u_j = 0, \qquad s = i\omega. \tag{3}$$

The boundary conditions on the outer surface $S_p^{\rm pl}$ (accordingly $S_p^{\rm el}$) are given by

$$N_i(\sigma_j^i + \mathring{\sigma}_\alpha^i \nabla^\alpha u_j) = p_j, \qquad u_j\Big|_{r\to\infty} \to 0. \tag{4}$$

In this case, $p_i = \mathring{p}_k \nabla^k u_j$ and $X_i = \mathring{X}_k \nabla^k u_j$ for "follower" loading and $p_i = X_i = 0$ for "dead" loading. Here and below, ∇ denotes covariant differentiation, the superscripts "el" and "pl" denote quantities that refer to the elastic or plastic region, respectively, and a circle at the top denotes the components of the basic unperturbed state.

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The continuity conditions on the elastoplastic boundary γ are given by

$$[N_i(\sigma^i_j + \mathring{\sigma}^i_\alpha \nabla^\alpha u_j)] = 0, \qquad [u_i] = 0. \tag{5}$$

For an incompressible viscoelastoplastic material in the case of an inhomogeneous basic state in the plastic and elastic regions, the relationship between the peak values of the stresses and displacements can be written as

$$\sigma_{ij} = \varepsilon_{\alpha\beta} a^\alpha_{ij} g^{\alpha\beta} + 2\mu\varepsilon_{ij} + \varepsilon_{12} a^4_{ij} + p g_{ij}. \tag{6}$$

The coefficients a^s_{ij} have the form

$$a^1_{ij} = a f_{ij}(-2f_{11} + r^2 f_{22})/3, \qquad a^2_{ij} = a f_{ij}(f_{11} - 2r^2 f_{22})/3, \qquad a^3_{ij} = a f_{ij}(f_{11} + r^2 f_{22})/3,$$
$$a^4_{ij} = -2a f_{ij} f_{12}, \qquad f_{ij} = S^0_{ij} - c\mathring{\varepsilon}^{\rm pl}_{ij}, \qquad a = 4\mu^2/(k^2(2\mu + c + \eta s)), \tag{7}$$

where p is the Lagrangian factor, $s = i\omega$ ($\omega = \alpha + i\beta$), and μ is the Lamé parameter. For $a = 0$, relations (6) and (7) correspond to the elastic region.

System (3)–(7) is a closed system of equations for stability problems in which there is an interface between the regions of elastic and plastic behavior of the material under loading.

A rock mass with a horizontal tunnel having a regular polygonal cross section (with rounded angles) will be modeled by an infinite weightless plate with a polygonal hole of radius R_B, whose contour is subjected to a uniformly distributed load q_0 (the fluid or gas pressure on the tunnel). The quantity of q_0 is such that the plastic region completely encompasses the tunnel contour. At infinity, the stresses in the plate tend to the quantity gh (g is the volumetric weight of the rock and h is the tunnel depth), i.e., the inherent stress in the rock mass (before tunnel boring) is considered hydrostatic.

In the determination of the components of the basic stress–strain states, all functions are written as series in powers of a small parameter δ that characterizes the deviation of an unperturbed state from the initial state, i.e., the deviation of a circle of radius R_0 from the regular polygon (B-gon) whose contour is given by

$$R_B = \sum_{n=0}^{\infty} \delta^n R^{(n)}_B = R_0\Big(1 + \delta\cos B\theta - \frac{3}{4}\,\delta^2 d'^2(1 - \cos 2B\theta + \ldots)\Big), \qquad 0 \leqslant \theta \leqslant 2\pi,$$

$$\{\sigma_{ij}, \varepsilon^{\rm pl}_{ij}, \varepsilon^{\rm el}_{ij}, e^{\rm pl}_{ij}, \ldots\} = \sum_{n=0}^{\infty} \delta^n \{\sigma^{(n)}_{ij}, \varepsilon^{\rm pl(n)}_{ij}, \varepsilon^{\rm el(n)}_{ij}, e^{\rm pl(n)}_{ij}, \ldots\}.$$

The zero approximation corresponds to the axisymmetric state of a plane with a circular hole of radius R_0, and in polar coordinates (r, θ), according to [8], it takes the following form:

— in the plastic region ($R_0 < r < 1$),

$$\sigma^{(0)}_r = -q_0 + \frac{4\chi\mu}{2\mu + c}\Big[\frac{c + 2\mu\,{\rm e}^{-\alpha t}}{4\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r^2}\Big) + (1 - {\rm e}^{-\alpha t})\ln\frac{r}{R_0}\Big],$$

$$\sigma^{(0)}_\theta = -q_0 + \frac{4\chi\mu}{2\mu + c}\Big[\frac{c + 2\mu\,{\rm e}^{-\alpha t}}{4\mu}\Big(\frac{1}{R_0^2} + \frac{1}{r^2}\Big) + (1 - {\rm e}^{-\alpha t})\Big(1 + \ln\frac{r}{R_0}\Big)\Big], \tag{8}$$

$$\varepsilon^{\rm pl(0)}_\theta = -\varepsilon^{\rm pl(0)}_r = \frac{\chi(1 - {\rm e}^{-\alpha t})}{2\mu + c}\Big(\frac{1}{r^2} - 1\Big),$$

where μ is the shear modulus, $\chi = {\rm sign}\,(q_0 - gh)$, and $\alpha = (2\mu + c)/\eta$;

— in the elastic range ($1 < r < \infty$),

$$\sigma^{(0)}_r = -gh - \frac{1}{r^2}, \qquad \sigma^{(0)}_\theta = -gh + \frac{1}{r^2}, \qquad \varepsilon^{\rm pl(0)}_\theta = -\varepsilon^{\rm pl(0)}_r = \frac{\chi}{2\mu r^2}. \tag{9}$$

The equation for the radius $\gamma^{(0)}$ of the elastoplastic boundary in the rock mass has the form

$$|q_0 - gh|(2\mu + c) - 2\mu + 4\mu\ln R_0(1 - {\rm e}^{-\alpha t}) - (2\mu\,{\rm e}^{-\alpha t} + c)/R_0^2 = 0. \tag{10}$$

According to [8], the first approximation is written as follows:

— in the plastic region ($R_0 < r < 1$),

$$\sigma_r^{(1)} = \frac{m_1}{2}\Big(\frac{1}{R_0^2} - \frac{1}{r^2} - 2\ln\frac{r}{R_0}\Big) + \frac{2AR_0d'}{r}\,(\sqrt{B^2-1}\sin\phi_1 - \cos\phi_1)\cos B\theta,$$

$$\sigma_\theta^{(1)} = \frac{m_1}{2}\Big(\frac{1}{R_0^2} + \frac{1}{r^2} - 2 - 2\ln\frac{r}{R_0}\Big) + \frac{2AR_0d'}{r}\,(\sqrt{B^2-1}\sin\phi_1 - \cos\phi_1)\cos B\theta,$$

$$\tau_{r\theta}^{(1)} = -\frac{2m_1AR_0d'}{r}\,\cos\phi_1'\sin B\theta, \tag{11}$$

$$\varepsilon_\theta^{\mathrm{pl}(1)} = \frac{m_1}{2\mu}\Big(1 - \frac{2a_0+1}{r^2}\Big) - \frac{1}{r}\Big\{B\sqrt{B^2-1}\,(c_1\sin\phi - c_2\cos\phi)$$

$$- \frac{AR_0d'}{\mu(2\mu+c)}\Big[((1+B^2)\cos\phi_1 - \sqrt{B^2-1}\,\sin\phi_1)\,\frac{\mu(1-\mathrm{e}^{-\alpha t})}{r^2}$$

$$+ \frac{B^2(2\,\mathrm{e}^{-\alpha t}+c)}{\sqrt{B^2-1}}(\sin\phi_1 + \sqrt{B^2-1}\,\ln r\cos\phi_1)\Big]\Big\}\sin B\theta,$$

$$\varepsilon_r^{\mathrm{pl}(1)} = -\varepsilon_\theta^{\mathrm{pl}(1)}.$$

Here

$$c_1 = \frac{AR_0d'}{B\mu(2\mu+c)(B^2-1)}\,\{\mu(1-\mathrm{e}^{-\alpha t})(B^2-1)\cos\phi_0$$

$$+ \sqrt{B^2-1}\,[\mu(m^2-1)(1-\mathrm{e}^{-\alpha t}) - B(2\mu+c)]\sin\phi_0\},$$

$$c_2 = \frac{AR_0d'}{B\mu(2\mu+c)(B^2-1)}\,\{\sqrt{B^2-1}\,[\mu(1-\mathrm{e}^{-\alpha t})(B^2+1) - m(2\mu+c)]\cos\phi_0$$

$$+ [B^2(2\mu\,\mathrm{e}^{-\alpha t}+c) + (1-\mathrm{e}^{-\alpha t})(B^2-1)\mu]\sin\phi_0\},$$

$$A = \frac{1}{2\mu+c}\Big[2\mu(1-\mathrm{e}^{-\alpha t}) + \frac{c+2\mu\,\mathrm{e}^{-\alpha t}}{R_0^2}\Big], \qquad m_1 = \frac{2c}{2\mu+c}\,(1-\mathrm{e}^{-\alpha t}) + 2\,\mathrm{e}^{-\alpha t},$$

$$\phi = \sqrt{B^2-1}\,\ln r, \quad \phi_1 = \sqrt{B^2-1}\,\ln\frac{r}{R_0}, \quad \phi_0 = \sqrt{B^2-1}\,\ln R_0, \quad a_0 = \frac{1}{2}\Big(\frac{1}{R_0^2} - 1 + 2\ln R_0\Big);$$

— in the elastic range ($1 < r < \infty$),

$$\sigma_r^{(1)} = \frac{m_1a_0}{r^2} - \frac{M}{2}\Big(\frac{B+2}{r^B} - \frac{B}{r^{B+2}}\Big) + N\Big(\frac{B+2}{r^{B+2}} - \frac{B+2}{r^B}\Big)\cos B\theta,$$

$$\sigma_\theta^{(1)} = -\frac{m_1a_0}{r^2} - \frac{M}{2}\Big(\frac{B}{r^{B+2}} - \frac{B-2}{r^B}\Big) + N\Big(\frac{B-2}{r^B} - \frac{B+2}{r^{B+2}}\Big)\cos B\theta, \tag{12}$$

$$\tau_{r\theta}^{(1)} = \frac{M}{2}\Big(\frac{B}{r^{B+2}} - \frac{B}{r^B}\Big) - N\Big(\frac{B}{r^B} - \frac{B+2}{r^{B+2}}\Big)\sin B\theta,$$

where $M = 2AR_0d'(\sqrt{m^2-1}\,\sin\phi_0 + \cos\phi_0)$ and $N = 2BAR_0d'\cos\phi_0$.

The equation for the radius $\gamma^{(1)}$ of the elastoplastic boundary has the form

$$\gamma^{(1)} = -\frac{(2\mu+c)m_1a_0}{4\mu(1-\mathrm{e}^{-\alpha t})} + \frac{2\mu+c}{2\mu(1-\mathrm{e}^{-2t})}\,BAR_0d'\cos\phi_0\cos B\theta. \tag{13}$$

In (8)–(13), all quantities having the dimension of stresses are related to the yield strength k, and those having the dimension of length to the radius $\gamma^{(0)}$ of the elastoplastic boundary in the unperturbed state.

To determine the zero and first approximations of this problem, we used the equilibrium equations, the plasticity state (1), the associate law of plastic flow (2), the relations linking the total elastic and plastic strains, the general equations of elastic theory, boundary conditions, and joining conditions for the solutions in the elastic and plastic regions.

Under the assumption of continuing loading [5] and incompressibility of the rock mass, Eqs. (3)–(7) are a closed system of equations for studying the stability of the basic state (8)–(13) of a horizontal tunnels with a polygonal cross section in the case where there is an interface between the regions of elastic and plastic behavior of the material in the loaded rock mass. This is a system of partial differential equations for the displacement vector component u, v, and w and the hydrostatic pressure p in the plastic and elastic zones of the rock mass. The nontrivial solution of this problem corresponds to the loss of stability of the basic state. To find the eigenvalues of the problem, we approximated the displacement and hydrostatic pressure in the zones of elastic and plastic deformation of the rock mass by double trigonometric series:

$$u = \sum_n^\infty \sum_m^\infty A_{nm}(r) \cos m\theta \cos nz, \qquad v = \sum_n^\infty \sum_m^\infty B_{nm}(r) \sin m\theta \cos nz,$$

$$w = \sum_n^\infty \sum_m^\infty C_{nm}(r) \cos m\theta \sin nz, \qquad p = \sum_n^\infty \sum_m^\infty D_{nm}(r) \cos m\theta \cos nz$$

(n and m are wave-formation parameters).

Substituting the functions u, v, and w, p into the linear stability equations (3) and taking into account (6) and (7) and the incompressibility condition, after a number of transformations, we obtain the following infinite system of ordinary differential equations for A_{nm} and B_{nm}:

$$\xi_1 A(r) + \xi_2 A'(r) + \xi_3 A''(r) + \xi_4 A'''(r) + \xi_5 A^{IV}(r) + \xi_6 B(r) + \xi_7 B'(r) + \xi_8 B''(r) + \xi_9 B'''(r) = 0,$$

$$\xi_{10} A(r) + \xi_{11} A'(r) + \xi_{12} A''(r) + \xi_{13} A'''(r) + \xi_{14} B(r) + \xi_{15} B'(r) + \xi_{16} B''(r) = 0. \tag{14}$$

Here

$$\xi_1 = \Big\{ a_{2,r} + \frac{1}{r}\,(a_{10,\theta} - a_6) - \frac{\sigma_\theta^0}{r}\,(1+m^2) + r\rho\omega^2 - m^2 a_{12} - n^2 \mu r$$

$$+ \frac{1}{r}\Big[a_7 - a_{11,\theta} - r a_{3,r} + \frac{1}{n^2}\Big(\frac{3\mu}{r^2}\,(1-m^2) - \frac{3m^2}{r^2}\,\sigma_\theta^0 + \frac{m^2}{r}\,\sigma_{\theta,r}^0$$

$$+ \frac{1}{r}\Big(r\rho\omega^2 - \frac{3}{r}\,(\tau_{r\theta,\theta}^0 - 3\sigma_r^0) - 5\sigma_{r,r}^0 + r\sigma_{r,rr}^0 + \tau_{r\theta,r\theta}^0 \Big)\Big)\Big]\Big\} \cos m\theta + \Big\{ a_4 - a_{12,\theta} + r a_{4,r} - a_8 - \tau_{r\theta,r}^0$$

$$+ \frac{1}{r}\Big[a_{11} - a_{10} - \sigma_{\theta,\theta}^0 + \frac{1}{n^2}\Big(\frac{6\tau_{r\theta}^0}{r^2} - \frac{4}{r}\,\tau_{r\theta,r}^0 - \frac{3}{r^2}\,\sigma_{\theta,\theta}^0 + \tau_{r\theta,r\theta}^0 + \frac{1}{r}\,\sigma_{\theta,\theta r}^0 \Big)\Big]\Big\} m \sin m\theta,$$

$$\xi_2 = \Big\{ a_1 + a_2 - 2a_3 - a_5 + a_7 + r(a_{1,r} - a_{3,r}) + a_{9,\theta} - a_{11,\theta} - \sigma_r^0 + r\sigma_{r,r}^0 + \tau_{r\theta,\theta}^0$$

$$+ \frac{1}{r}\,(\lambda+\mu) - \frac{1}{n^2}\Big[\frac{\mu}{r^2}\,(3+m^2) + \frac{m^2}{r^2}\,\sigma_\theta^0 - \frac{m^2}{r}\,\sigma_{\theta,r}^0$$

$$+ \frac{1}{r}\Big(r\rho\omega^2 + r\sigma_{r,rr}^0 - \frac{3}{r}\,\tau_{r\theta,\theta}^0 + \frac{9}{r}\,\sigma_r^0 + \tau_{r\theta,r\theta}^0 - 5\sigma_{r,r}^0 \Big)\Big]\Big\} \cos m\theta$$

$$+ \Big\{ r a_4 + a_{11} - a_9 - 2\tau_{r\theta}^0\Big(1 + \frac{3}{r^2 n^2}\Big) + \frac{1}{n^2}\Big[\frac{2\tau_{r\theta,r}^0}{r} - \frac{1}{r^2}\,\sigma_{\theta,\theta}^0 + \tau_{r\theta,r\theta}^0 + \frac{1}{r}\,\sigma_{\theta,\theta r}^0 \Big]\Big\} m \sin m\theta,$$

$$\xi_3 = \Big\{ r(a_1 - a_3 + \sigma_r^0) + (r-1)(\lambda+\mu)$$

$$- \frac{1}{n^2}\Big[r\rho\omega^2 - \frac{\mu}{r}\,(m^2+3) - \frac{m^2}{r}\,\sigma_\theta^0 + \sigma_{r,r}^0 - \frac{3}{r}\,\sigma_r^0 + \tau_{r\theta,r\theta}^0 + r\sigma_{r,rr}^0 \Big]\Big\} \cos m\theta + \Big\{ \frac{1}{r}\,\sigma_{\theta,\theta r}^0 + 3\tau_{r\theta,r}^0 \Big\} \frac{m}{n^2}\,\sin m\theta,$$

$$\xi_4 = -\frac{1}{n^2}\,\{2r\sigma^0_{r,r} + 2\mu + \tau^0_{r\theta,\theta}\}\cos m\theta + \frac{2m}{n^2}\,\tau^0_{r\theta}\sin m\theta, \qquad \xi_5 = -\frac{r(\mu+\sigma^0_r)}{n^2}\,\cos m\theta,$$

$$\xi_6 = \Big\{a_{2,r} - a_{12} + \frac{1}{r}\Big[a_{10,\theta} - a_6 - 2\sigma^0_\theta - a_{11,\theta} + a_7 - ra_{3,r}$$

$$-\frac{1}{n^2}\Big(\frac{\mu}{r^2}\,(5+3m^2) + \frac{3m^2}{r^2}\,\sigma^0_\theta - \frac{m^2}{r}\,\sigma^0_{\theta,r} + \rho\omega^2 - \frac{3}{r}\,\sigma^0_{r,r} - \frac{1}{r^2}\,\tau^0_{r\theta,\theta}$$

$$+\frac{7}{r^2}\,\sigma^0_r - \sigma^0_{r,rr} - \frac{1}{r}\,\tau^0_{r\theta,r\theta}\Big)\Big]\Big\}m\cos m\theta + \Big\{ra_{4,r} - a_{12,\theta} + a_4 - a_8 - \tau^0_{r\theta,r} - \frac{1}{r}\,(\sigma^0_{\theta,\theta} + m^2(a_{10}-a_{11}))$$

$$-\frac{m^2}{n^2 r}\Big[\frac{2}{r^2}\,\tau^0_{r\theta} + \frac{4}{r}\,\tau^0_{r\theta,r} + \frac{3}{r^2}\,\sigma^0_{\theta,\theta} - \tau^0_{r\theta,r\theta} - \frac{1}{r}\,\sigma^0_{\theta,\theta r}\Big]\Big\}\sin m\theta,$$

$$\xi_7 = \Big\{a_2 + ra_{12} - a_3 - \frac{1}{rn^2}\Big[r\rho\omega^2 + \frac{\mu}{r}\,(3-m^2) - \frac{m^2}{r}\,\sigma^0_\theta + r\sigma^0_{r,rr}$$

$$-\frac{3}{r}\,\tau^0_{r\theta,\theta} + \frac{9}{r}\,\sigma^0_r - 5\sigma^0_{r,r} + \tau^0_{r\theta,r\theta}\Big]\Big\}m\cos m\theta$$

$$+\Big\{r\Big(a_{12,\theta} - a_4 + a_8 - ra_{4,r} - \frac{2}{r}\,\tau^0_{r\theta}\Big) - \frac{m^2}{n^2 r}\Big(\frac{6}{r}\,\tau^0_{r\theta} - 3\tau^0_{r\theta,r} - \frac{1}{r}\,\sigma^0_{\theta,\theta}\Big)\Big\}\sin m\theta,$$

$$\xi_8 = -\frac{m}{rn^2}\Big\{2r\sigma^0_{r,r} - 2\mu - 4\sigma^0_r + \tau^0_{r\theta,\theta}\Big\}\cos m\theta + \Big\{\frac{2m^2}{rn^2}\,\tau^0_{r\theta} - r^2 a_4\Big\}\sin m\theta,$$

$$\xi_9 = -\frac{m(\mu+\sigma^0_r)}{n^2}\,\cos m\theta, \tag{15}$$

$$\xi_{10} = m\Big\{a_{8,\theta} - 2a_{12} - a_{12,r} + \frac{1}{r}\Big[a_7 - a_6 - 2\sigma^0_\theta + \frac{1}{rn^2}\Big(r\rho\omega^2 + \frac{\mu}{r}\,(1-m^2)$$

$$-\frac{m^2}{r}\sigma^0_\theta - \sigma^0_{r,r} - \frac{1}{r}\,(\tau^0_{r\theta,\theta} - 3\sigma^0_r)\Big)\Big]\Big\}\cos m\theta\sin m\theta$$

$$+\Big\{\frac{1}{r}\Big[(a_{10}-a_{11})\Big(2-\frac{1}{r}\Big) + a_{10,r} + a_{6,\theta} + r\tau^0_{r\theta,r} + \sigma^0_{\theta,\theta} - a_{11,r} - a_{7,\theta} + rm^2 a_8\Big]\Big\}\cos^2 m\theta$$

$$+\Big(\frac{m}{rn}\Big)^2\Big\{\frac{1}{r}\,(2\tau^0_{r\theta} - \sigma^0_{\theta,\theta}) - \tau^0_{r\theta,r}\Big\}\sin^2 m\theta,$$

$$\xi_{11} = m\Big\{a_7 - a_{12} - a_5 + \frac{1}{r}\Big[(\lambda+\mu)(1-r)$$

$$+\frac{1}{n^2}\Big(r\rho\omega^2 - \frac{\mu}{r}\,(1+m^2) - \frac{m^2}{r}\,\sigma^0_\theta + \sigma^0_{r,r} + \frac{1}{r}\,(\tau^0_{r\theta,\theta} - 3\sigma^0_r)\Big)\Big]\Big\}\cos m\theta\sin m\theta$$

$$+\Big\{2\tau^0_{r\theta} + 2a_9 + a_{9,r} + \frac{1}{r}\,a_{10,r} + a_{5,\theta} - a_{11,r} - a_{7,\theta} - a_{11}\Big(2+\frac{1}{r}\Big)\Big\}\cos^2 m\theta$$

$$-\frac{m^2}{rn^2}\Big\{\frac{1}{r}\,(2\tau^0_{r\theta} + \sigma^0_{\theta,\theta}) + \tau^0_{r\theta,r}\Big\}\sin^2 m\theta,$$

$$\xi_{12} = \frac{m}{rn^2}\,\{2\mu + r\sigma^0_{r,r} + \tau^0_{r\theta,\theta}\}\cos m\theta\sin m\theta + \{a_9 - a_{11}\}\cos^2 m\theta - \frac{2m^2}{rn^2}\,\tau^0_{r\theta}\sin^2 m\theta,$$

$$\xi_{13} = \frac{m}{n^2}\,\{\mu + \sigma^0_r\}\cos m\theta\sin m\theta,$$

$$\xi_{14} = \Big\{a_{8,\theta} - 2a_{12} - a_{12,r} + r\rho\omega^2\Big(1+\Big(\frac{m}{rn}\Big)^2\Big) - n^2\mu r - \frac{1}{r}\,\sigma^0_\theta$$

$$+\frac{m^2}{r}\Big[a_7-a_6-\sigma_\theta^0+\frac{1}{rn^2}\Big(\frac{\mu}{r}(1-m^2)-\frac{m^2}{r}\,\sigma_\theta^0-\sigma_{r,r}^0-\frac{1}{r}\,(\tau_{r\theta,\theta}^0-3\sigma_r^0)\Big)\Big]\Big\}\cos m\theta\sin m\theta$$

$$+\frac{m}{r}\Big\{(a_{10}-a_{11})\Big(2-\frac{1}{r}\Big)+a_{10,r}+r\tau_{r\theta,r}^0+\sigma_{\theta,\theta}^0+a_{6,\theta}-a_{11,r}-a_{7,\theta}+ra_8\Big\}\cos^2 m\theta$$

$$-\frac{m^3}{r^3n^2}\,\{2\tau_{r\theta}^0-\sigma_{\theta,\theta}^0-r\tau_{r\theta,r}^0\}\sin^2 m\theta,$$

$$\xi_{15}=\Big\{r\Big(2a_{12}+a_{12,r}-a_{8,\theta}+\sigma_{r,r}^0+\frac{1}{r}\,(\tau_{r\theta,\theta}^0-\sigma_r^0)\Big)+\Big(\frac{m}{rn}\Big)^2(\tau_{r\theta,\theta}^0+r\sigma_{r,r}^0-3\sigma_r^0-\mu)\Big\}\cos m\theta\sin m\theta$$

$$+m\Big\{\frac{1}{r}\,(a_{10}-a_{11})-ra_8+2\tau_{r\theta}^0\Big\}\cos^2 m\theta-\frac{2m^3}{r^2n^2}\,\tau_{r\theta}^0\sin^2 m\theta,$$

$$\xi_{16}=\frac{m}{n^2}\Big\{r(a_{12}+\sigma_r^0)+\frac{m^2}{rn^2}\,(\mu+\sigma_r^0)\Big\}\cos m\theta\sin m\theta.$$

In this case, in the plastic region $V^{\rm pl}$ in the rock mass, the precritical state is defined by formulas (8) and (11), and in the elastic range $V^{\rm el}$, it is defined by formulas (9) and (12). For simplicity, in (14) and below, the subscripts n and m at the quantities A and B are omitted.

For $r=R_0(1+\delta\cos B\theta-(3/4)\delta^2d'^2(1-\cos 2B\theta+\ldots))$ $(0\leqslant\theta\leqslant 2\pi)$, boundary conditions (4) on the inner contour of the tunnel, in view of (6) and (7), become

$$A\varphi_1+A'\varphi_2+A''\varphi_3+A'''\varphi_4+B\varphi_5+B'\varphi_6+B''\varphi_7=0,$$

$$A\varphi_8+A'\varphi_9+B\varphi_{10}+B'\varphi_{11}=0, \qquad (16)$$

$$A\varphi_{12}+A'\varphi_{13}+A''\varphi_{14}+B\varphi_{15}+B'\varphi_{16}=0,$$

where

$$\varphi_1=-\frac{1}{r}\Big\{a_3-a_2-\mu+\frac{1}{rn^2}\Big[r\rho\omega^2+\frac{\mu}{r}(1-m^2)-\frac{m^2}{r}\,\sigma_\theta^0+\frac{1}{r}\,(3\sigma_r^0-\tau_{r\theta,\theta}^0-r\sigma_{r,r}^0)\Big]\Big\}\cos m\theta$$

$$+m\Big\{a_4-\frac{1}{r}\,\tau_{r\theta}^0\Big(1+\frac{2}{r^2n^2}\Big)+\frac{1}{n^2r^2}\Big(\tau_{r\theta,r}^0+\frac{1}{r}\sigma_{\theta,\theta}^0\Big)\Big\}\sin m\theta,$$

$$\varphi_2=\Big\{a_1-a_3+\sigma_r^0-\frac{1}{r}\Big[(1-r)(\lambda+\mu)+\frac{1}{n^2}\Big(r\rho\omega^2-\frac{\mu}{r}\,(m^2+1+n^2r^2)$$

$$-\frac{m^2}{r}\,\sigma_\theta^0+\frac{1}{r}\,(r\sigma_{r,r}^0-3\sigma_r^0+\tau_{r\theta,\theta}^0)\Big)\Big]\Big\}\cos m\theta+\frac{m}{rn^2}\Big\{\frac{1}{r}\,\sigma_{\theta,\theta r}^0+\tau_{r\theta,r}^0+\frac{2}{r}\,\tau_{r\theta}^0\Big\}\sin m\theta,$$

$$\varphi_3=-\frac{1}{rn^2}\,\{r\sigma_{r,r}^0+2\mu+\tau_{r\theta,\theta}^0\}\cos m\theta+\frac{2m}{rn^2}\,\tau_{r\theta}^0\sin m\theta,\qquad \varphi_4=-\frac{\mu+\sigma_r^0}{n^2}\,\cos m\theta,$$

$$\varphi_5=-\frac{m}{r}\Big\{a_3-a_2-\mu+\frac{1}{rn^2}\Big[r\rho\omega^2+\frac{\mu}{r}\,(1-m^2)-\frac{m^2}{r}\,\sigma_\theta^0+\frac{1}{r}\,(3\sigma_r^0-\tau_{r\theta,\theta}^0-r\sigma_{r,r}^0)\Big]\Big\}\cos m\theta$$

$$+\Big\{a_4-\frac{1}{r}\,\tau_{r\theta}^0+\Big(\frac{m}{nr}\Big)^2\Big(\tau_{r\theta,r}^0+\frac{1}{r}\,\sigma_{\theta,\theta}^0-\frac{2}{r}\,\tau_{r\theta}^0\Big)\Big\}\sin m\theta,$$

$$\varphi_6=\frac{m}{r^2n^2}\,\{\mu+3\sigma_r^0-r\sigma_{r,r}^0-\tau_{r\theta,\theta}^0\}\cos m\theta+\Big\{2\Big(\frac{m}{rn}\Big)^2\tau_{r\theta}^0-ra_4\Big\}\sin m\theta, \qquad (17)$$

$$\varphi_7=-\frac{m(\mu+\sigma_r^0)}{rn^2}\cos m\theta,\qquad \varphi_8=\frac{1}{r}\,\{a_{10}-a_{11}+\tau_{r\theta,\theta}^0\}\cos m\theta-ma_{12}\sin m\theta,$$

$$\varphi_9=\{a_9-a_{11}\}\cos m\theta,\qquad \varphi_{10}=\frac{m}{r}\,\{a_{10}-a_{11}+\tau_{r\theta,\theta}^0\}\cos m\theta-a_{12}\sin m\theta,$$

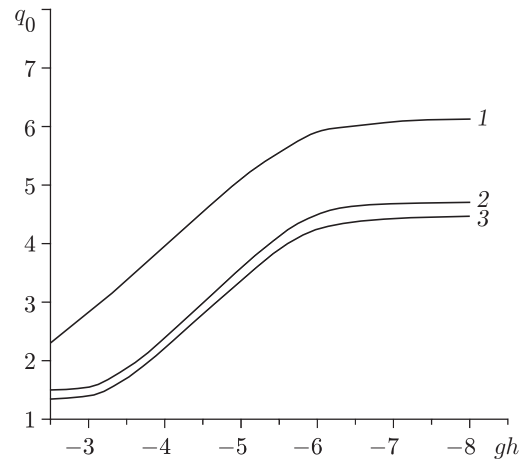

Fig. 1. Critical pressure on the tunnel contour versus hydrostatic pressure gh ($\eta = 0.001$) for $c = 0.9$ (1), 0.1 (2), and 0.01 (3).

$$\varphi_{11} = \{ra_{12} + \sigma_r^0\}\sin m\theta, \qquad \varphi_{12} = \Big\{n\mu - \frac{\mu + \sigma_r^0}{nr^2}\Big\}\cos m\theta - \frac{m}{nr^2}\,\tau_{r\theta}^0 \sin m\theta,$$

$$\varphi_{13} = \frac{\mu + \sigma_r^0}{nr}\cos m\theta - \frac{m}{nr}\,\tau_{r\theta}^0 \sin m\theta, \qquad \varphi_{14} = \frac{\mu + \sigma_r^0}{nr}\cos m\theta,$$

$$\varphi_{15} = -\frac{m(\mu + \sigma_r^0)}{nr^2}\cos m\theta - \frac{m^2}{nr^2}\,\tau_{r\theta}^0 \sin m\theta, \qquad \varphi_{16} = \frac{m(\mu + \sigma_r^0)}{nr}\cos m\theta.$$

In view of (6) and (7), the stress continuity conditions (5) on the elastoplastic boundary $\gamma = \gamma^{(0)} + \delta\gamma^{(1)}$ ($0 \leqslant \theta \leqslant 2\pi$) are written as

$$A^{\rm pl}\zeta_1 + A'^p\zeta_2 + A''^p\varphi_3^{\rm pl} - A''^e\varphi_3^{\rm el} + A'''^p\varphi_4^{\rm pl} - A'''^p\varphi_4^{\rm pl} + B^{\rm pl}\zeta_5 + B'^p\zeta_6 + B''^p\varphi_7^{\rm pl} - B''^e\varphi_7^{\rm el} = 0,$$
$$A^{\rm pl}\zeta_8 + A'^p\zeta_9 + B^{\rm pl}\zeta_{10} + B'^p\zeta_{11} = 0, \tag{18}$$
$$A^{\rm pl}\zeta_{12} + A'^p\zeta_{13} + A''^p\varphi_{14}^{\rm pl} - A''^e\varphi_{14}^{\rm el} + B^{\rm pl}\zeta_{15} + B'^p\zeta_{16} = 0,$$

where $\zeta_i = \varphi_i^{\rm pl} - \varphi_i^{\rm el}$ $(i = 1, 2, \ldots, 16)$.

The condition of localization of the perturbations $u_j \to 0$ as $r \to \infty$ ($j = 1, 2, 3$) implies that

$$(A')^{\rm el} = 0, \qquad (A'')^{\rm el} = 0, \qquad (B')^{\rm el} = 0, \qquad (B'')^{\rm el} = 0. \tag{19}$$

Since it is not possible to find the exact analytical solution of the boundary-value problem (14)–(19), we seek an approximate solution using the finite difference method [9]. The method is based on the replacement of the derivatives of the functions $A(r)$ and $B(r)$ by finite-difference expressions. As a result, we obtain a homogeneous system of algebraic equations linear in the parameters A_{nm} and B_{nm}. From this it follows that the determination of the critical load q_0 corresponding to local buckling of a horizontal tunnel with a polygonal cross section reduces to solving a matrix equation. In the calculation of the determinant, along with finding the basic stress–strain state for each region $V^{\rm pl}$, $V^{\rm el}$ of the rock mass (8), (11), (9), (12), it is necessary to take into account Eqs. (10) and (13), which define the position of the elastoplastic boundary γ in the rock mass. Minimization should be performed for the difference-grid size, the wave-formation parameters along the contour m and the generatrix n, and the material and design parameters λ_j. Thus, we obtain the problem of multidimensional optimization of the quantity q_0 as a function of m and n provided that the determinant of the resulting algebraic system is equal to zero: $\det(q_0, m, n, \lambda_j) = 0$.

The calculations were performed for the case where the rock mass contained a tunnel having a cross section in the shape of a square ($B = 4$) with rounded angles. Figures 1–3 show the critical pressure on the tunnel contour versus hydrostatic pressure gh. It was assumed in this case that $R_0 = 0.4$, $\delta = 0.06$, and $\mu = 1$ and the wave-formation parameters $n = m = 4$.

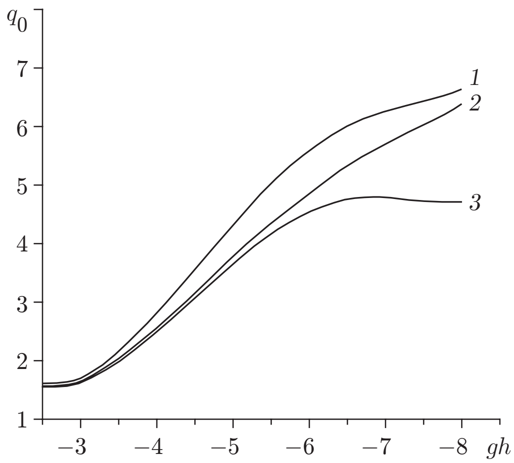

Fig. 2

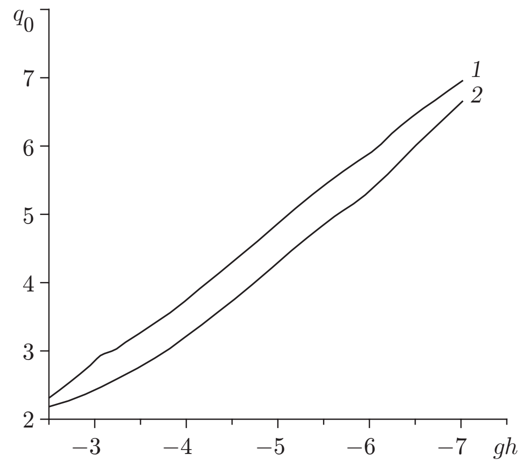

Fig. 3

Fig. 2. Critical pressure on the tunnel contour versus gh for $c = 0.1$ and $\eta = 0.001$ (1), 0.01 (2), and 0.1 (3).

Fig. 3. Critical pressure on the tunnel contour versus gh for $c = 0.9$ and $\eta = 0.001$: curves 1 and 2 refer to a tunnel in the shape of a regular tetragon with rounded angles ($B = 4$) and circle ($B = 60$), respectively.

An analysis of the numerical experiment shows that:

— The critical pressure on the tunnel contour increases with increase in the depth of tunnel location (see Figs. 1–3);

— the critical pressure on the tunnel contour increases with increase in the hardening coefficient c (see Fig. 1);

— the buckling load on the tunnel contour decreases with increase in the viscosity; in this sense, it is possible to speak of the stabilizing role of viscosity in the medium (see Fig. 2);

— the stability region is larger for a circular cylindrical tunnel than for a tunnel with a square cross section (see Fig. 3).

Setting $\delta = 0$ in relations (8)–(13), we arrive at the results obtained in [4] for a circular cylindrical tunnel.

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